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Mountain Pass Characterization of Least Energy Solutions and its Application

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0. Introduction

This note is based on my joint works [**JT1**, **JT2**, **JT3**, **JT4**] with L. Jeanjean and we consider the following nonlinear elliptic problem:

$$\begin{aligned} -\Delta u &= g(u) \qquad \text{in } \mathbf{R}^N, \\ u &\in H^1(\mathbf{R}). \end{aligned} \tag{0.1}$$

Here $g : \mathbf{R} \to \mathbf{R}$ is a continuous function. Problem (0.1) and similar problems in a bounded domain appear in various problems in mathematical physics etc.

We are mainly interested in least energy solutions of (0.1). A solution $u_0 \in H^1(\mathbf{R}^N)$ of (0.1) is said to be a *least energy solution* if it satisfies

$$I(u_0) = m,$$

where

$$m = \inf\{I(u);\, u \neq 0,\ u(x) \text{ is a solution of } (0.1)\}. \tag{0.2}$$

Here $I(u) \in C(H^1(\mathbf{R}^N), \mathbf{R})$ is a functional corresponding to (0.1), that is,

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2\, dx - \int_{\mathbf{R}^N} G(u)\, dx, \\ G(s) &= \int_0^s g(\tau)\, d\tau. \end{aligned} \tag{0.3}$$

The main purpose of this note is to give a characterization of least energy solutions through mountain pass theorem and give an application to a singular perturbation problem for nonlinear Schrödinger type equations.

1. Mountain pass characterization of least energy solutions

First we recall so-called Mountain Pass Theorem. Let E be a Hilbert space and $I(u) \in C^1(E, \mathbf{R})$. We say that $I(u)$ has a *mountain pass geometry* if it has the following properties:

(i) $I(0) = 0$.

(ii) There exist $\rho_0 > 0$, $\delta_0 > 0$ such that

$$I(u) \geq \delta_0 \quad \text{for all } \|u\|_E = \rho_0.$$

(iii) There exists $u_0 \in E$ such that

$$\|u_0\|_E > \rho_0 \quad \text{and} \quad I(u_0) < 0.$$

For a function $I(u)$ with mountain pass geometry we can define the following minimax value (Mountain Passs value):

$$b = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(u)),$$

where

$$\Gamma = \{\gamma(t) \in C([0,1], E);\ \gamma(0) = 0,\ I(\gamma(1)) < 0\}.$$

Our main question is the following:

Question: For a functional $I(u) \in C^1(H^1(\mathbf{R}^N), \mathbf{R})$ defined in (0.3), does Mountain Pass Theorem give a least energy solution? In other words, does it hold

$$b = m? \tag{1.1}$$

Here m is defined in (0.2).

Remark 1.1. (i) If $I(u) \in C^1(E, \mathbf{R})$ satisfies the Palais-Smale compactness condition, then $b > 0$ is a critical value of $I(u)$ by the Mountain Pass Theorem. That is, there exists $u_0 \in E$ such that $I(u_0) = b$ and $I'(u_0) = 0$.
(ii) For a functional $I(u)$ defined in (0.3), working in the space $H^1_r(\mathbf{R}^N)$ of radially symmetric functions, we can get some compactness. However under the conditions (g0)–(g3) below, we don't know whether $I(u)$ satisfies the Palais-Smale condition or not.

The standard way to insure (1.1) so far is to assume that

$$s \mapsto \frac{g(s)}{s} : (0, \infty) \to \mathbf{R} \text{ is non-decreasing.} \tag{1.2}$$

We remark that under suitable conditions in addition to (1.2) the above property (1.2) we can make use of the *Nehari manifold*: $\mathcal{M} = \{u \in H^1(\mathbf{R}^N) \setminus \{0\};\ I'(u)u = 0\}$ and we can get a least energy solution through minimizing problem: $\inf_{u \in \mathcal{M}} I(u)$.

Our first theorem ensures that (1.1) holds without assumption (1.2).

Theorem 1.2. ([**JT1**]) *Assume* $N \geq 2$ *and*

(g0) $g(s) \in C(\mathbf{R}, \mathbf{R})$ *is continuous and odd.*

(g1) $-\infty < \liminf_{s \to 0} \frac{g(s)}{s} \leq \limsup_{s \to 0} \frac{g(s)}{s} < 0$ *for* $N \geq 3$,

$\lim_{s \to 0} \frac{g(s)}{s} \in (-\infty, 0)$ *for* $N = 2$.

(g2) When $N \geq 3$, $\lim_{s \to \infty} \frac{|g(s)|}{s^{\frac{N+2}{N-2}}} = 0$.

When $N = 2$, *for any* $\alpha > 0$ *there exists* $C_\alpha > 0$ *such that*

$$|g(s)| \leq C_\alpha e^{\alpha s^2} \quad \textit{for all } s \geq 0.$$

(g3) There exists $s_0 > 0$ *such that* $G(s_0) > 0$.

Then $I(u)$ *given in* (0.3) *has a mountain pass geometry and* (1.2) *holds. Moreover for any least energy solution* $\omega(x)$ *of* (0.1) *there exists a path* $\gamma \in \Gamma$ *such that*

$$\gamma(x) \in \gamma([0,1]) \quad \textit{and} \quad \max_{t \in [0,1]} I(\gamma(t)) = I(\omega). \tag{1.3}$$

Remark 1.3. Under (g0)–(g3), it is shown in Berestycki-Lions [**BL**] (for $N \geq 3$) and Berestycki-Gallouët-Kavian [**BGK**] (for $N = 2$) that $m > 0$ and the existence of least energy solutions. We remark that (g0)–(g3) are almost necessary conditions for the existence of solutions (see [**BL**] and [**BGK**]).

When $N = 1$, we have the following result.

Theorem 1.4. ([**JT4**]) *Suppose* $N = 1$ *and assume (g0), (g1) and*

(g3') There exists $s_0 > 0$ *such that*

$$\begin{aligned} G(s) &< 0 \quad \textit{for all } s \in (0, s_0), \\ G(s_0) &= 0, \\ g(s_0) &> 0. \end{aligned}$$

Then (0.1) *has a unique solution* $\omega(x)$ *up to translation and it has a mountain pass characterization, that is,*

$$I(\omega) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

Remark 1.5. When $N = 1$, conditions (g0), (g1), (g3') are necessary and sufficient for the existence of solutions of (0.1).

Here we explain an idea of the proof of Theorem 1.2 just for $N \geq 3$. We make use of properties of the dilation $u_t(x) = u(x/t)$ $(t > 0)$ as in **[BL]** and **[BGK]**. Actually for any least energy solution $\omega(x)$ of (0.1), a path defined by

$$\gamma(t) = \begin{cases} \omega(x/t) & t > 0, \\ 0 & t = 0 \end{cases} \tag{1.4}$$

gives a continuous path in $H^1(\mathbf{R}^N)$ and

$$I(u) = \frac{t^{N-2}}{2} \int_{\mathbf{R}^N} |\nabla u|^2 \, dx - t^N \int_{\mathbf{R}^N} G(\omega) \, dx. \quad \text{for all } \ t \geq 0.$$

Thus we can see that $I(\gamma(t)) \to -\infty$ as $t \to \infty$ and $\tilde{\gamma}(t) = \gamma(Lt)$ satisfies (1.3) for large $L > 1$. In particular, it ensures $b \leq m$. To show $b \geq m$, we introduce the set of non-trivial functions satisfying Pohozaev identity:

$$\mathcal{P} = \{u \in H^1(\mathbf{R}^N) \setminus \{0\};\ \frac{N-2}{N} \int_{\mathbf{R}^N} |\nabla u|^2 \, dx - N \int_{\mathbf{R}^N} G(u) \, dx = 0\}.$$

We can show

(i) $m = \inf_{u \in \mathcal{P}} I(u)$,
(ii) $\gamma([0,1]) \cap \mathcal{P} \neq \emptyset$ for all $\gamma \in \Gamma$.

$b \geq m$ easily follows from the above 2 properties.

Remark 1.6. When $N = 1$, 2, the situation is a little bit different. For example, a path given in (1.4) is not continuous at $t = 0$. So we need further aruguments. See **[JT1, JT4]**.

Remark 1.7. Under the condition (1.2), we can see easily that a path define by $\gamma(t) = tL\omega(x)$ $(L \gg 1)$ satisfies $\gamma \in \Gamma$ and (1.3).

2. An application to a singular perturbation problem

Mountain Pass characterization of least energy solutions is useful in various situations. Here we give an application in a singular perturbation problem.

We consider the existence of positive solutions of nonlinear Schödinger equations:

$$\begin{aligned} &-\epsilon^2 \Delta u + V(x) u = f(x) \quad \text{in } \mathbf{R}^N, \\ &u \in H^1(\mathbf{R}^N), \end{aligned} \tag{2.1}$$

where $f(s) \in C^1(\mathbf{R}, \mathbf{R})$ and $V(x) : \mathbf{R}^N \to \mathbf{R}$ is a Hölder continuous function satisfying

(V) $\inf_{x \in \mathbf{R}^N} V(x) > 0$.

We try to find a family of solutions $u_\epsilon(x)$ concentrating around a given local minimum of the potential $V(x)$ as $\epsilon \to 0$. This problem is studied in various situations. See [**ABC**, **DF1**, **DF2**, **DFT**, **FW**, **Gr**, **Gu**, **KW**, **YYL**, **NT1**, **NT2**, **O1**, **O2**, **P**, **R**, **W**] and references therein.

If we introduce a rescaled (around $x_0 \in \mathbf{R}^N$) function $v(y) = u_\epsilon(\epsilon y + x_0)$, equation (2.1) becomes

$$-\Delta v + V(\epsilon y + x_0)v = f(v) \quad \text{in } \mathbf{R}^N.$$

Taking a limit as $\epsilon \to 0$, it appears an autonomous problem:

$$-\Delta v + V(x_0)v = f(v) \quad \text{in } \mathbf{R}^N. \tag{2.2}$$

(2.2) is very important is the study of (2.1). For example, if ground state solutions of the limit equation (2.2) are unique and non-degenerate, we can apply a Lyapunov-Schmidt reduction method to find a family of concentrating solutions. See [**FW**, **O1**, **O2**, **ABC**, **YYL**, **Gr**, **P**].

In what follows, we argue without assumption of uniqueness and non-degeneracy of solutions of (2.2). We take a variational approach, which was first done by Rabinowitz [**R**] and developed considerably by del Pino-Felmer [**DF1**]. Mountain pass characterization of least energy solutions for (2.2), which is a conclusion of our Theorem 1.2, is very helpful and it enables us to deal with asymptotically linear equations as well as superlinear ones.

To state our result, we need the following assumptions:

(f0) $f(s) \in C^1(\mathbf{R}, \mathbf{R})$.

(f1) $f(x) = o(s)$ as $s \sim 0$.

(f2) For some $p \in (1, \frac{N+2}{N-2})$ if $N \geq 3$ and for some $p \in (0, \infty)$ if $N = 1, 2$

$$\frac{f(s)}{s^p} \to 0 \quad \text{as} \quad s \to \infty.$$

Our main result is the following

Theorem 2.1. ([**JT3**]) *Suppose $N \geq 2$ and assume (V), (f0)–(f2) and one of the following 2 conditions:*

(f3) There exists $\mu > 2$ such that

$$0 < \mu \int_0^s f(\tau)\, d\tau \leq f(s)s \quad \textit{for all } s > 0.$$

(f4) $s \mapsto \frac{f(s)}{s}$; $(0, \infty) \to \mathbf{R}$ is non-decreasing.

Let $\Lambda \subset \mathbf{R}^N$ be a bounded open set satisfying

$$\inf_{x \in \Lambda} V(x) < \min_{x \in \partial\Lambda} V(x) \tag{2.3}$$

and, if $a \equiv \lim_{s\to\infty} \frac{f(s)}{s} < \infty$ *under the assumption (f4), we assume moreover that*

$$\inf_{x\in\Lambda} V(x) < a.$$

Then there exists an $\epsilon_0 > 0$ *such that for* $\epsilon \in (0, \epsilon_0]$, (2.1) *has a solution* $u_\epsilon(x)$ *satisfying*

1° $u_\epsilon(x)$ *has unique local maximum (hence global maximum) in* $\mathbf{R}^N$ *at* $x_\varepsilon \in \Lambda$.

2° $V(x_\varepsilon) \to \inf_{x\in\Lambda} V(x)$.

3° *There exist constants* C_1, $C_2 > 0$ *such that*

$$u_\epsilon(x) \le C_1 \exp\left(-C_2 \frac{|x - x_\varepsilon|}{\varepsilon}\right) \quad \text{for } x \in \mathbf{R}^N.$$

Remark 2.2. (i) Condition (f3) is called Ambrosetti-Rabinowitz' superlinear growth condition and it implies

$$f(s) \ge Cs^{\mu-1} \quad \text{for all } s \ge 1.$$

In particular, it implies $\frac{f(s)}{s} \to \infty$ as $s \to \infty$ and $f(s)$ has a superlinear growth.
(ii) Condition (f4) does not require superlinear growth of $f(s)$. In particular, we can deal with a class of asymptotically linear equations. For example,

$$f(s) = \frac{s^2}{1+s}$$

satisfies (f0)–(f2) and (f4).

Remark 2.3. (f4) can be generalized to the following condition (f5):

(f5) (i) There exists $a \in (0, \infty]$ such that

$$\frac{f(\xi)}{\xi} \to a \quad \text{as } \xi \to \infty.$$

(ii) There exists a constant $D \ge 1$ such that

$$\widehat{F}(s) \le D\widehat{F}(t) \quad \text{for all } 0 \le s \le t,$$

where

$$\widehat{F}(\xi) = \frac{1}{2} f(\xi)\xi - F(\xi).$$

It is easily observed that (f4) implies (f5) with $D = 1$. We remark that the condition (f5) is due to Jeanjean [**J**], in which the existence of positive solutions for asymptotically linear elliptic problems is considered. In particular, boundedness of Palais-Smale sequences is obtained under (f5) via concentration-compactness type argument. We also refer to [**JT1**] for asymptotically linear elliptic problems.

Remark 2.4. We remark that our result give a generalization of the result of del Pino-Felmer [**DF1**], in which a family of solutions $u_\epsilon(x)$ is found under conditions (V), (f0)–(f2) and both of (f3) and (f4).

When $N = 1$, the existence of solution concentrating in a bounded open set $\Lambda \subset \mathbf{R}$ satisfying (2.3) can be shown under weaker conditions, namely, under (f0), (f1) and the following condition:

(f6) There exists $\xi_0 > 0$ such that

$$\begin{aligned} &-\frac{\sigma}{2}\xi^2 + F(\xi) < 0 \quad \text{for } \xi \in (0, \xi_0), \\ &-\frac{\sigma}{2}\xi_0^2 + F(\xi_0) = 0, \\ &-\sigma\xi_0 + f(\xi_0) > 0, \end{aligned}$$

where $\sigma = \inf_{x\in\Lambda} V(x)$.

For the proof we follow the argument in [**DFT**] where broken geodesic type argument is developed for 1-dimensional nonlinear Schrödinger equations. We can also construct solutions with clustering spikes as in [**DFT**].

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